CONSTRUCTION OF PBIB DESIGNS FROM FAMILIES OF DIFFERENCE SETS

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1. Introduction

Vartak [3] introduced Rectangular Association Scheme PBIB designs (RAS-PBIBDs) into the literature on statistical designs. He arrived at and obtained these designs by the application of the Kronecker product of matrices procedure to incidence matrices of various designs. This method of construction of designs has the drawback that the parameter values of the design obtained increases too fast as the parameter values of the original design or designs used for the construction becomes larger and larger. This is particularly disturbing when it comes to block size. In this paper a few Rectangular Association Scheme and two-associate class PBIB designs are constructed using the well known method of differences. For details of the Rectangular Association Scheme and other association schemes with two-associate classes we refer to [2]. Many of the designs constructed herein can be used for two-way elimination of heterogeneity.

2. Some Results

We now present the construction of a few Rectangular type PBIB designs. Since designs with $r>10$ are not of much practical interest we restrict overselves to the construction of designs with $r\leqslant 10$.

The details of the proofs are omitted.

Theorem 2·1. The RAS−PBIBDs having parameters

$$v=3(2t+1),\quad b=3t\,(2t+1),\quad r=4t,\quad k=4,$$
$$\lambda_1=t,\quad \lambda_2=1,\quad \lambda_3=2,$$
$$n_1=2,\quad n_2=2t,\quad n_3=4t,$$

where $t=1, 2$ can always be constructed.

Proof. Consider residue classes mod $(2t+1)$, $t=1, 2$. Let the elements of this module be denoted by $0, 1, 2, \ldots, 2t$. To each element u of the module let there correspond three symbols u_1, u_2 and u_3. Now consider the $3t$ sets:

$$\begin{array}{l}(1_1\ (2t)_1\ 0_2\ 0_3),\ (2_1\ (2t-1)_1\ 0_2\ 0_3),\ \ldots,\ (t_1\ (t+1)_1\ 0_2\ 0_3)\\ (1_2\ (2t)_2\ 0_3\ 0_1),\ (2_2\ (2t-1)_2\ 0_3\ 0_1),\ \ldots,\ (t_2\ (t+1)_2\ 0_3\ 0_1)\\ (1_3\ (2t)_3\ 0_1\ 0_2),\ (2_3\ (2t-1)_3\ 0_1\ 0_2),\ \ldots,\ (t_3\ (t+1)_3\ 0_1\ 0_2)\end{array} \quad \ldots(2.1)$$

Arrange the treatments as

$$\begin{array}{ccc} 0_1 & 0_2 & 0_3 \\ 1_1 & 1_2 & 1_3 \\ \cdots & \cdots & \cdots \\ (2t)_2 & (2t)_2 & (2t)_3 \end{array} \quad \ldots(2.2)$$

in rows and columns. Then it can be verified with the help of corresponding initial sets for BIBD given in [2] that in the sets developed from (2.1) in mod $(2t+1)$ any two treatments in the same row of (2.2) occur t times, any two treatments in the same column of (2.2) occur once and two treatments not in the same row or column of (2.2) occur twice. Hence the theorem.

Theorem 2·2. The RAS-PBIBDs having parameters

$$\begin{array}{lll} v=6t+3, & b=3t\,(2t+1), & r=5t, \\ k=5, & \lambda_1=3t, & \lambda_2=3, \\ \lambda_3=2, & n_1=2, & n_2=2t, \\ n_3=4t, & & \end{array}$$

where $t=1, 2$ can always be constructed.

Proof. Let $S_{11}, \ldots, S_{1t}$; $S_{21}, \ldots, S_{2t}$; $S_{31}, \ldots, S_{3t}$ denote the iditial sets of (2.1) in the order given therein. Further if $s=(s_1\ s_2\ \ldots\ s_k)$, let (s/u) denote the set $(s_1\ s_2\ \ldots\ s_k u)$. Then the sets $(s_{11}/0_1)$, $(s_{12}/0_1)$, ..., $(s_{1t}/0_1)$; $(s_{21}/0_2)$, $(s_{22}/0_2)$, ..., $(s_{2t}/0_2)$; $(s_{31}/0_3)$, $(s_{32}/0_3)$, ..., $(s_{3t}/0_3)$; ...(1.3)

can be verified to generate the required designs with the association scheme (2.2) when developed in mod $(2t+1)$, $t=1, 2$.

Theorem 2·3. The RAS-PBIBs having parameters $v=3\,(4t+1)$, $b=3t\,(4t+1)$, $r=5t$, $k=5$, $\lambda_1=0$, $\lambda_2=1$, $\lambda_3=2$, $n_1=2$, $n_2=4t$, $n_3=8t$, where $t=1, 2$ can always be constructed.

Proof. Let x denote a primitive root of the Galois Field GF $(4t+1)$, $t=1, 2$. To each element u of GF $(4t+1)$ let there correspond three varieties u_1, u_2, u_3. Consider the $3t$ sets:

$$\left(x_1^{2i} \quad x_1^{2t+2i} \quad x_2^{\alpha+2i} \quad x_2^{\alpha+2t+2i} \quad 0_3 \right),$$
$$\left(x_2^{2i} \quad x_2^{2t+2i} \quad x_3^{\alpha+2i} \quad x_3^{\alpha+2t+2i} \quad 0_1 \right),$$
$$\left(x_3^{2i} \quad x_3^{2t+2i} \quad x_1^{\alpha+2i} \quad x_1^{\alpha+2t+2i} \quad 0_2 \right) \qquad \ldots(2.4)$$

where $i=0$ for $t=1$ and $i=0$, 1 for $t=2$, and where we take α to be odd, say $\alpha=1$. Then if we write down the treatments in rows and columns as:

$$\begin{array}{ccc} 0_1 & 0_2 & 0_3 \\ x_1^0 & x_2^0 & x_3^0 \\ \cdots & \cdots & \cdots \\ x_1^{4t-1} & x_2^{4t-1} & x_3^{4t-1} \end{array} \qquad \ldots(2.5)$$

and if we take the association relation as indicated in the proof of Theorem 2.1, it can be verified with the help of corresponding sets given for BIBD in [2] that these sets when developed in GF $(4t+1)$ lead to the required designs.

Theorem 2·4. The RAS-PBIBDs having parameters $v=6t+3$, $b=3t\,(2t+1)$, $r=5t$, $k=5$, $\lambda_1=0$, $\lambda_2=2$, $\lambda_3=4$, $n_1=2$, $n_2=2t$, $n_3=4t$, where $t=1$, 2 can always be constructed.

Proof. Let x denote a primitive root of GF $(2t+1)$, $t=1$, 2. To each element u of GF $(2t+1)$ let there correspond three varieties u_1, u_2, u_3.

Then consider the sets:

$$\left(x_1^{i} \quad x_2^{t+i} \quad x_2^{\alpha+i} \quad x_2^{\alpha+i+t} \quad 0_3 \right),$$
$$\left(x_2^{i} \quad x_2^{t+i} \quad x_3^{\alpha+i} \quad x_3^{\alpha+i+t} \quad 0_1 \right),$$
$$\left(x_3^{i} \quad x_3^{t+i} \quad x_1^{\alpha+i} \quad x_1^{\alpha+i+i} \quad 0_2 \right), \qquad \ldots(2.6)$$

where $i=0$ for $t=1$ and $i=0$, 1 for $t=2$ and $0<\alpha\neq t$. Now, arrange the treatments in rows and columns as:

$$\begin{array}{ccc} 0_1 & 0_2 & 0_3 \\ x_1^0 & x_2^0 & x_3^0 \\ \cdots & \cdots & \cdots \\ x_1^{2t-1} & x_2^{2t-1} & x_3^{2t-1} \end{array} \qquad \ldots(2.7)$$

and consider the association relation between treatments as in the proof of Theorem 2.1. Then it can be verified with the help of corresponding sets for BIBD given in [2] that the sets in (2.6) when developed in mod (2t+1), $t=1$, 2 lead to the required designs.

Theorem 2·5. The RAS-PBIBD having parameters $v=b=9$, $r=k=6$, $\lambda_1=1$, $\lambda_2=4$, $\lambda_3=5$, $n_1=2$, $n_2=2$, $n_3=4$ can be constructed.

Proof. Let the treatments be defined as in Theorem 2.4 for $t=1$. Consider the sets

$$\left(x_1^0\, x_1\, x_2^\alpha \quad x_2^{\alpha+1}\, 0_3\, 0_1 \right), \quad \left(x_2^0\, x_2\, x_3^\alpha \quad x_3^{\alpha+1}\, 0_1\, 0_2 \right),$$

$$\left(x_3^0\, x_3\, x_1^\alpha \quad x_1^{\alpha+1}\, 0_2\, 0_3 \right).$$

Then by developing the sets in (2.8) in mod (3) and defining the association relation between treatments as in connection with (2.7) we obtain the required design.

Theorem 2·6. The RAS-PBIBDs having parameters $v=3\,(2t+1)$, $b=3t\,(2t+1)$, $r=5t$, $k=5$, $\lambda_1=0$, $\lambda_2=4$, $\lambda_3=3$, $n_1=2$, $n_2=2t$, $n_3=4t$, where $t=1$, 2 can always be constructed.

Proof. Let the treatments be difined as in Theorem 2.4. Consider the sets.

$$\left(x_1^i \quad x_1^{t+i}\, x_2^{\alpha+i} \quad x_2^{\alpha+t+i} \quad 0_1 \right), \left(x_2^i \quad x_2^{t+i}\, x_3^{\alpha+i}\, x_3^{\alpha+t+i}\, 0_2 \right),$$

$$\left(x_3^i \quad x_3^{t+i}\, x_1^{\alpha+i}\, x_1^{\alpha+t+i} \quad 0_3 \right),$$

where $i=0$ for $t=1$ and $i=0$, 1 for $t=2$. If we arrange the treatments as in (2.7) and define the association relation between treatments ae therein, we get the PBIB designs of the Theorem from these initial sets.

Theorem 2·7. We can always construct the RAS-PBIEDs having parameters $v=6t+3$, $b=(2t+1)\,(2t+u)$, $r=(2t+u)$, $k=3$, $\lambda_1=u$, $\lambda_2=0$, $\lambda_3=1$, $n_1=2$, $n_2=2t$, $n_3=4t$, where t, u, $0<t\leqslant 4$, $0<u\leqslant 8$, are integers.

Proof. Consider the module, residue classes mod $(2t+1)$, $t=1$, 2, 3, 4. Let the elements of the module be 0, 1, 2, ..., $2t$. Let to each element u of this module, there correspond three varieties u_1, u_2, u_3. Consider the sets

$$(1_1\ (2t)_2\ 0_3),\ (2_1\ (2t-1)_2\ 0_3),\ \ldots,\ (t_1(t+1)_2\ 0_3),\ (1_2\ (2t)_1\ 0_3),$$
$$(2_2\ (2t-1)_1\ 0_3),\ \ldots,\ (t_2\ (t+1)_1\ 0_3),\ (0_1\ 0_2\ 0_3),\ \ldots,\ (0_1\ 0_2\ 0_3). \quad \ldots(2.10)$$

the set $(0_1\ 0_2\ 0_3)$ occurring in (2.10) u times. It can then be verified that these sets form a family of difference sets for the design given in the theorem, when the treatments are arranged and association relation between treatments are defined as indicated in the proof of Theorem 2.1.

Corollary 2.7.1. When $u=1$ in the above theorem, the RAS-PBIBDs therein reduces to Semi-regular Group Divisible designs.

Corollary 2.7.2. RAS-PBIBDs with parameters $v=6t+3$, $b=2t\ (2t+1)$, $r=2t$, $k=3$, $\lambda_1=\lambda_2=0$, $\lambda_3=1$, $n_1=2$, $n_2=2t$. $n_3=4t$, where t, $0<t\leqslant 5$, is an integer, can always be constructed.

Proof. Omit sets $(0_1\ 0_2\ 0_3)$ from (2.10). The remaining initial sets will generate this design.

Theorem 2.8. We can always construct RAS-PBIBDs having parameters $v=6t+3$, $b=(4t+u)\ (2t+1)$, $r=4t+u$, $k=3$, $\lambda_1=u$, $\lambda_2=0$, $\lambda_3=2$, $n_1=2$, $n_2=2t$, $n_3=4t$, where t, u, $0<t\leqslant 2$, $0<u\leqslant 6$, are integers.

Proof. Consider mod $(2t+1)$, $t=1, 2$. To each element u of this module let there correspond three varieties u_1, u_2, u_3. Now consider the sets

$$(1_1\ (2t)_2\ 0_3),\ (2_1\ (2t-1)_2\ 0_3),\ \ldots,\ (t_1\ (t+1)_2\ 0_3)\ ;$$
$$(1_2\ (2t)_1\ 0_3),\ (2_2\ (2t-1)_1\ 0_3),\ \ldots,\ (t_2\ (t+1)_1\ 0_3);$$
$$(1_1\ (2t)_3\ 0_2),\ (2_1\ (2t-1)_3\ 0_2),\ \ldots,\ (t_1\ (t+1)_3\ 0_2);$$
$$(1_3\ (2t)_1\ 0_2),\ (2_3\ (2t-1)_1\ 0_2),\ \ldots,\ (t_3\ (t+1)_1\ 0_2):$$
$$(0_1\ 0_2\ 0_3),\ \ldots,\ (0_1\ 0_2\ 0_3), \quad \ldots(2.11)$$

where the set $(0_1\ 0_2\ 0_3)$ is repeated u times. If we arrange the treatments as in (2.2) and also define the association relation between treatments as therein, we see that the sets in (2.11) give rise to the designs of the theorem.

Corollary 2.8.1. The RAS-PBIBDs with parameters $v=6t+3$, $b=4t\ (2t+1)$, $r=4t$, $k=3$, $\lambda_1=\lambda_2=0$, $\lambda_3=2$. $n_1=2$, $n_2=2t$, $n_3=4t$, where $t=1, 2$ can always be constructed.

For, we obtain these by omitting the set $(0_1\ 0_2\ 0_3)$ from (2.11).

Theorem 2.9. It is possible to construct the R AS-PBIBDs having parameters $v=9$, $b=3\ (6+u)$, $r=6+u$, $k=3$, $\lambda_1=u$, $\lambda_2=0$, $\lambda_3=3$, $n_1=n_2=2$, $n_3=4$, where u, $0<u\leqslant 4$, is an integer.

Proof. Define the treatments as in the proof of Theorem 2.8 for $t=1$. Then consider the sets:

$$(1_1\ 2_2\ 0_3),\ (1_2\ 2_1\ 0_3),\ (1_1\ 2_3\ 0_2),\ (1_3\ 2_1\ 0_2),\ (1_2\ 2_3\ 0_1);$$
$$(1_3\ 2_2\ 0_1),\ (0_1\ 0_2\ 0_3),\ \ldots,\ (0_1\ 0_2\ 0_3), \qquad \ldots(2.12)$$

where the set $(0_1\ 0_2\ 0_3)$ occurs in (2.12) u times. The proof is exactly similar to that of Theorem 2.8.

Theorem 2.10. The RAS-PBIBDS having parameters $v=9$, $b=18$, $r=10$, $k=5$, $\lambda_1=6$, $\lambda_2=4$, $\lambda_3=5$, $n_1=n_2=2$, $n_3=4$ can be constructed.

Proof. Let the treatments be defined as in the proof of Theorem 2.8 for $t=1$. Consider the initial sets

$$(1_1\ 2_2\ 0_3\ 0_2\ 0_1),\ (1_2\ 2_1\ 0_3\ 0_1\ 0_2),\ (1_1\ 2_3\ 0_2\ 0_3\ 0_1),$$
$$(1_3\ 2_1\ 0_2\ 0_1\ 0_3),\ (1_2\ 2_3\ 0_1\ 0_3\ 0_2),\ (1_3\ 2_2\ 0_1\ 0_2\ 0_3) \qquad \ldots(2.13)$$

Proceeding as in the proof of the preceding theorem we can prove this result.

Theorem 2.11. The Rectangular Association Scheme PBIBD with parameters $v=b=15$, $r=k=9$, $\lambda_1=0$, $\lambda_2=4$, $\lambda_2=8$, $n_1=4, n_2=2$, $n_3=8$ can be constructed.

Proof. Consider mod (3). To each element u of this field let there correspond five varieties u_1, u_2, u_3, u_4, u_5. Let x be a primitive element of GF (3), and consider the 5 sets:

$$\left(x_1^0\ x_1\ x_3^\alpha\ x_3^{\alpha+1}\ 0_2\ x_5\ x_5^2\ x_4^{\alpha+1}\ x_4^{\alpha+2}\right),$$
$$\left(x_2^0\ x_2\ x_4^\alpha\ x_4^{\alpha+1}\ 0_3\ x_1\ x_1^2\ x_5^{\alpha+1}\ x_5^{\alpha+2}\right),$$
$$\left(x_3^0\ x_3\ x_5^\alpha\ x_5^{\alpha+1}\ 0_4\ x_2\ x_2^2\ x_1^{\alpha+1}\ x_1^{\alpha+2}\right),$$
$$\left(x_4^0\ x_4\ x_1^\alpha\ x_1^{\alpha+1}\ 0_5\ x_3\ x_3^2\ x_2^{\alpha+1}\ x_2^{\alpha+2}\right),$$
$$\left(x_5^0\ x_5\ x_2^\alpha\ x_2^{\alpha+1}\ 0_1\ x_4\ x_4^2\ x_3^{\alpha+1}\ x_3^{\alpha+2}\right), \qquad \ldots(2.14)$$

Arranging the treatments in raws and columns as

0_1	0_2	0_3	0_4	0_5
x_1^0	x_2^0	x_3^0	x_4^0	x_5^0
x_1	x_2	x_3	x_4	x_5

and defining the association relation between these treatments just like in the case of (2.2) it can be verified with the help of corresponding difference sets given for BIBD in [2] that (2.14) constitutes a family of difference sets for the design of this theorem.

3. Concluding Remarks

It is to be noted the designs we have obtained herein are from the initial family of sets used to generate BIB designs. This shows that BIB designs can be used to construct Rectangular PBIB designs in a way different from the Kronecker product method.

A second thing worth noting is the idea of designs with elastic blocks. We may define this class of designs as the one, which is closed under the operarion of enlargement or reduction of block sizes. It is worth noting that designs obtained in Theorems 2.2 and 2.5 are as a result of block enlargement of designs obtained in Theorems 2.1 and 2.4 respectively.

In this paper we have given many families of initial sets involving suffixed symbols which lead to difference set solutions to certain RAS-PBIB designs. Most of these families have the important property explained below.

If one takes the family of initial sets (2.1) of the paper, for example, one can see that treatment symbols having distinct suffixes 1, 2 and 3 accur equally frequently in the first position of all the sets of the family. This is true for all the other positions as well. Since the blocks are generated by addition modulo the appropriate integers and the suffixes are kept unchanged during the addition process, the above fact ensures that all the treatments occur equally frequently in each of the positions of the blocks. Thus one has a RAS-PBIB design in which in each position of the blocks all treatments occur equally frequently. For example, consider sets (2.1) for $t=1$. The initial sets are:

$$(1_1\ 2_1\ 0_2\ 0_3),\ (1_2\ 2_2\ 0_3\ 0_1),\ (1_3\ 2_3\ 0_1\ 0_2).$$

These when developed in mod (3) give the design (columns are blocks)

1_1	2_1	0_1	1_2	2_2	0_2	1_3	2_3	0_3
2_1	1_2	1_1	2_2	0_2	1_2	2_3	0_3	1_3
0_2	1_2	2_2	0_3	1_3	2_3	0_1	1_1	2_1
0_3	1_3	2_3	0_1	1_1	2_1	0_2	1_2	2_2

It can be seen that all the treatments occur exactly once in each of the four positions (rows) of the blocks. Thus this RAS-PBIB can be used as a two-way elimination of heterogeneity design.

Thus one can see that the RAS-PBIB disigns given in Theorems 2.1, 2.2, 2.3, 2.4, 2.5, 2.6. 2.10 and 2.11 can be used as two-way elimination of heterogeneity designs straightway, whereas those given in Theorems 2.7, 2.8 and 2.9 can be used in this way if $(2t+u)$, $(4t+u)$ and u respectively therein are divisible by 3, those given in coroliary 2.7.2 when $2t \equiv 0 \mod (3)$ and those given in corollary 2.8.1 if $4t \equiv 0 \mod (3)$. The semi-regular designs of corollary 2.7.1. can be used in this way if $2t+1 \equiv 0 \mod (3)$. A large number of the designs are, therefore, useful for two-way elimination of heterogeneity.

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